MATH 312
Lecture 29

THE JORDAN FORM

$$\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$$

Recap:
- Eigenvalues, $\det(A - \lambda I) = 0$.
- Eigenvectors: v in $N(A-\lambda I)$, $v \neq 0$.
- Diagonalizability, $A = SDS^{-1}$, good for "A^k"!
- "Bad" examples $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, two copies of eigenvalue 2, only ONE independent eigenvector ☹

Let's revisit the eigenstuff process: A is $n \times n$,

$\boxed{\det(A - \lambda I) = 0}$ is a POLYNOMIAL of degree n, and (using complex numbers if necessary) we can factor it:

$$(\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k} = 0.$$

Def Here, the matrix A has m_j copies of λ_j as an eigenvalue: m_j is called the "Algebraic multiplicity" of the eigenvalue λ_j

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Def The dimension $\dim N(A - \lambda_j I)$ is called the "Geometric multiplicity" of the eigenvalue λ_j, we will denote this by l_j

Now, we always have:

$$m_j \geqslant \ell_j \quad \text{and} \quad \underbrace{m_1 + m_2 + \cdots + m_k = n}_{\text{degree of } \det|A-\lambda I|=0.}$$

$\boxed{\text{But note}}$:

- If $m_j > n_j$ strictly, then we can <u>NOT</u> diagonalize. This is what happens with the matrix $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.

- $\det|A - \lambda I| = 0 \iff (3-\lambda)(1-\lambda)+1 = 0$
$$\iff \lambda^2 - 4\lambda + 4 = 0$$
$$\iff (\lambda-2)^2 = 0$$

Now, Algebraic multiplicity of the eigenvalue 2 is 2! We have "2 copies" of it.

- Unfortunately, $(A-2I) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and it only has 1D nullspace, spanned by the eigenvector $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Meaning, the geometric multiplicity of the eigenvalue 2 is only 1.

- So, $\underset{\text{mult}}{\text{geom}} < \underset{\text{mult}}{\text{alg}}$, and A is NOT diagonalizable. So,

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Q | HOW TO COMPUTE A^k FOR LARGE k's?

The REMEDY is fascinating in itself, so let's take some time to set things up. BASICALLY, the real question is "How close does a non-diagonalizable matrix come to being diagonalizable"??

Def A "JORDAN BLOCK" is a square matrix of the form

$$\begin{bmatrix} a & 1 & & 0 \\ & a & \ddots & \\ & & \ddots & 1 \\ 0 & & & a \end{bmatrix},$$

eg. $[2]$, $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, ...

i.e., same "a" on the diagonal, always a "1" above the diagonal entry, and zeros in all other spots

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Def A matrix is said to be "IN JORDAN FORM" if it looks like

$$J = \begin{bmatrix} J_1 & & O \\ & \ddots & \\ O & & J_k \end{bmatrix}$$

where each $J_\bullet$ is a Jordan Block.

eg

$$\left[\begin{array}{cc|c|ccc} 3 & 1 & 0 & & O & \\ 0 & 3 & 0 & & & \\ \hline 0 & 0 & 2 & 0 & 0 & 0 \\ \hline & & & -1 & 0 & 0 \\ & O & & 0 & -1 & 0 \\ & & & 0 & 0 & -1 \end{array}\right] \text{ and } \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{or maybe} \quad \left[\begin{array}{c|c|cc} 4 & 0 & & O \\ \hline 0 & 4 & & \\ \hline & & 4 & 1 \\ O & & 0 & 4 \end{array}\right]$$

Thm (Jordan's Theorem) Let A be an $n \times n$ matrix. Then,

- There is an invertible matrix S so that
$$A = SJS^{-1},$$
where J is in Jordan Form: $J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix}$

- Each block $J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$ has a <u>single eigenvalue</u> λ_i of A along the diagonal.

- There are as many Jorden blocks for λ_i as there are independent eigenvectors [so, # blocks = geom. multiplicity]

- There are $\underline{m_i}$ copies of λ_i in J.

- The matrix J is unique, given A. That is, <u>No</u> <u>OTHER</u> matrix $J' \neq J$ in Jordan form satisfies $A = PJ'P^{-1}$ for invertible P !!
(except permutations of Jordan Blocks...)

The proof, sadly, is beyond our reach right now. <u>BUT</u>, we can compute Jordan forms easily! Let's see what we need:

<u>Want</u>
$$A = SJS^{-1}, \quad \text{for } A \ n \times n.$$
$$\text{so } AS = SJ.$$

If S has the columns $[u_1, u_2, \ldots, u_n]$, then

$$A \begin{bmatrix} u_1 & \ldots & u_n \end{bmatrix} = \begin{bmatrix} u_1 & \ldots & u_n \end{bmatrix} \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix}$$

looks bad...

What <u>helps</u> here is to look at a <u>SMALL</u> example.

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

We already know: eigenvalues $\lambda_1 = 2$, occurs with algebraic multiplicity 2. It has <u>ONLY</u> one eigenvector (up to independence), $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

<u>So,</u> (By Jordan's Theorem) $A = SJS^{-1}$ with $J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

$\boxed{\text{WHAT is } S?}$

Again, $AS = SJ$ for $S = [\vec{a}\ \vec{b}]$.

Now, $A[\vec{a}\ \vec{b}] = [\vec{a}\ \vec{b}] \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$= [2\vec{a}\quad \vec{a} + 2\vec{b}]$.

Meaning: $\boxed{A\vec{a} = 2\vec{a}}$ and $\boxed{A\vec{b} = \vec{a} + 2\vec{b}}$

So, $\vec{a}$ is just the already known eigenvector, i.e.,

$$\vec{a} = \vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

THIS SIDE IS NEW!

$A\vec{b} - 2\vec{b} = \vec{a}$,

or $(A - 2I)\vec{b} = \vec{a}$

or, $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Many solutions exist (1D family), so choose something:

eg $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Putting this together,

$$A = SJS^{-1}$$

where $S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

ALGORITHM

Input: A, $n \times n$
Output J, S so that $A = SJS^{-1}$...

I Compute eigenvalues and eigenvectors of A:
let's say you find $\{\lambda_i\}_1^p$ with algebraic mult. $\{m_i\}_1^p$ and geometric mult. $\{\ell_i\}_1^p$

II Collect the eigenvalues in J:

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix}$$

There are as many blocks as indep eigenvectors, so $k = \ell_1 + \ell_2 + \cdots + \ell_p$.

So, $J_1 = \begin{bmatrix} \lambda_1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_1 \end{bmatrix}$ has size $(m_1 - \ell_1 + 1) \times (m_1 - \ell_1 + 1)$

III Compute generalized eigenvectors whenever a block has dimension > 1: eg $J_1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, compute

$$(A - 2I)v = u, \text{ where } u = \text{eigenvector for } 2.$$

TIME for A HARDER Example:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$ Want S, J with $A = SJS^{-1}$.

STEP I :

$$\det(A - \lambda I) = 0$$

$$\lambda(\lambda-2)^2 = 0$$

$$\Leftrightarrow \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ -1 & -\lambda & -1 \\ 1 & 2 & 3-\lambda \end{bmatrix} = 0$$

$$\Leftrightarrow (1-\lambda) \begin{vmatrix} -\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} -1 & -\lambda \\ 1 & 2 \end{vmatrix} = 0$$

$$\Leftrightarrow (1-\lambda)\left(\lambda^2 - 3\lambda + 2\right) + (-2+\lambda) = 0$$

Since $\lambda^2 - 3\lambda + 2$ factorizes to $(\lambda-2)(\lambda-1)$, we get

$$\Leftrightarrow -(\lambda-1)^2(\lambda-2) + (\lambda-2) = 0$$

$$\Leftrightarrow (\lambda-2)\left[-(\lambda-1)^2 + 1\right] = 0$$

So $\underline{\underline{\lambda = 2}}$ OR $(\lambda-1)^2 = 1$

So! $\underline{\underline{\lambda = 2}}$ or $\underline{\underline{\lambda = 0}}$

So : Eigenvalues of A are :

$\lambda_1 = 2$ with algebraic mult. $= 2$

$\lambda_2 = 0$ " " " $= 1$.

Eigenvectors : For $\lambda_1 = 2$, $N(A - 2I) = \begin{bmatrix} -1 & 0 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ has a 1D nullspace, spanned by $\vec{v_1} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

And for $\lambda_2 = 0$, $(A - 0I)$ itself has a 1D Nullspace, spanned by $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \vec{v}_2$

STEP II. Since Alg multiplicity of λ_2 is DIFFERENT from its Geom. multiplicity, it gets a Jordan Block of the form $\begin{bmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{bmatrix}$ ...

But $\lambda_2 = 0$ is okay!

So, $$J = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

Block for $\lambda_1 = 2$

Block for $\lambda_2 = 0$.

STEP III Already have $S = \begin{bmatrix} 1 & & 0 \\ -1 & & 1 \\ 1 & & -1 \end{bmatrix}$

$\vec{v}_1$ $\vec{v}_2$

Just need this column

The missing column is of course the generalized eigenvector for $\lambda_1 = 2$! It is some $\vec{u}$ which solves

$$(A - 2I)\vec{u} = \vec{v}_1, \text{ or}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ -1 & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Again, MANY solutions. I like $\vec{u} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

And FINALLY, we get

$$A = SJS^{-1},$$

where $J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

and $S = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$